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Gonihedric string equation I

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Abstract: We discuss the basic properties of the gonihedric string and the problem of its formulation in the continuum. Possible solution of the problem is given in the form of generalized Dirac equation and of the corresponding gamma matrices. The wave function and the Dirac matrices are infinite-dimensional. The spectrum of the theory consists of particles and antiparticles of increasing half-integer spin lying on quasi-linear trajectories of different slope. Explicit formulas for the mass spectrum allow to compute the string tension and thus demonstrate the string character of the theory. The string tension varies from trajectory to trajectory and indicates nonperturbative origin of the spectrum. The equation does not admit tachyonic solutions, but still has unwanted ghost solutions. The equation has its own value independent of the motivation advocated in this article.

Keywords: 2D Gravity, Lattice Gauge Field Theories.

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Contents

1. Introduction

There is some experimental and theoretical evidence for the existence of a string theory in four dimensions which may describe strong interactions and represent the solution of QCD [1].

One of the possible candidates for that purpose is the gonihedric string which has been defined as a model of random surfaces with an action which is proportional to the linear size of the surface [2]

$$A(M) = m \sum_{<ij>} \lambda_{ij} \cdot \Theta(\alpha_{ij}), \qquad \Theta(\alpha) = |\pi - \alpha|^{\varsigma}, \tag{1.1}$$

where λ_{ij} is the length of the edge $< ij >$ of the triangulated surface M and α_{ij} is the dihedral angle between two neighbouring triangles of M sharing a common edge

$< ij >$. The angular factor Θ defines the rigidity of the random surfaces [2] and the convergence of the partition function [2, 3, 4].

The model has a number of properties which make it very close to the Feynman path integral for a point-like relativistic particle. This can be seen from (1.1) in the limit when the surface degenerates into a single world line, in that case the action became proportional to the length of the path and classical equation of motion for the gonihedric string is reduced to the classical equation of motion for a free relativistic particle. The other important property of the theory is that at the classical level the string tension is equal to zero and quarks viewed as open ends of the surface are propagating freely without interaction [2] $\sigma_{classical} = 0$. This is because the gonihedric action (1.1) is equal to the perimeter $P(M_2)$ of the flat Wilson loop $A(M_2) \rightarrow P(M_2) = R + T$ and the potential $V(R)$ is constant. As it was demonstrated in [2], quantum fluctuations generate the nonzero string tension

$$\sigma_{quantum} = \frac{d}{a^2}(1 - ln\frac{d}{\beta})\,, \tag{1.2}$$

where d is the dimension of the spacetime, β is the coupling constant, a is the scaling parameter and $\varsigma = (d-2)/d$ in (1.1).[1]. The gonihedric string may consistently describe asymptotic freedom and confinement, as it is expected to be the case in QCD, and we have to ask what type of equation may describe this string . The aim of this article is to suggest a *possible* answer. Important hint comes from the analysis of the transfer matrix which has been constructed for this theory [7]. This transfer matrix describes the propagation of the closed contour M_1^τ in time direction τ with an amplitude which is proportional to the sum of the length of the contour and of its total curvature [7]

$$A(M_1^\tau) = \int_{M_1^\tau} dL + \int_{M_1^\tau} k(L) dL$$

and of the interaction which is proportional to the overlapping length of the contour on two neighbouring time slices

$$A_{int}(M^\tau; M^{\tau+1}) = \int_{M^\tau \cap\ M^{\tau+1}} dL.$$

Considering the system of closed contours on a given time slice τ as a separate system with the length and curvature amplitude one can see that this two-dimensional system has a continuum limit which can be described by a free Dirac fermion [7, 8]. Thus on every time slice the system generates a fermionic string which may propagate in time because of the week interaction between neighbouring time slices [7]. Thus the physical

[1] In addition to the formulation of the theory in the continuum space the system allows an equivalent representation on Euclidean lattices where a surface is associated with a collection of plaquettes [5, 6, 7, 9] The Monte Carlo simulations [10] demonstrate that the gonihedric system undergoes the second order phase transition and the string tension is generated by quantum fluctuations, as it was expected theoretically [2] (1.2).

picture of the fermionic string propagation which follows from the transfer matrix approach hints that the desired equation should describe propagation of fermionic degrees of freedom destributed over the space contour. When this contour shrinks to a point, the equation should describe propagation of a free fermion.

It is natural therefore to think that each particle in this theory should be viewed as a state of a complex fermionic system and that this system should have a point-particle limit when there is no excitation of the internal motion. The question is then how to incorporate this internal motion into existing point particle equations. Ettore Majorana suggested in 1932 [12] an extension of the Dirac equation by constructing an infinite-dimensional representation of the Lorentz group and the corresponding extension of the gamma matrices. The equation has the Dirac form [11, 12, 14]

$$\{i\Gamma_{\mu}\partial^{\mu} - M\}\Psi = 0 \tag{1.3}$$

and the Majorana commutation relations which define the Γ_{μ} matrices are given by the formula (see (13) in [12])

$$[\Gamma_{\mu}, I_{\lambda\rho}] = \eta_{\mu\lambda}\Gamma_{\rho} - \eta_{\mu\rho}\Gamma_{\lambda}\,, \tag{1.4}$$

where $I_{\mu\nu}$ are the generators of the Lorentz algebra. These equations allow to find the Γ_{μ} matrices when the representation of the $I_{\mu\nu}$ is given. The original Majorana solution for Γ_{μ} matrices is infinite-dimensional (see equation (14) in [12]) and the mass spectrum of the theory is equal to

$$M_j = \frac{M}{j + 1/2}\,, \tag{1.5}$$

where $j = 1/2, 3/2, 5/2, \ldots.$ in the fermion case and $j = 0, 1, 2, \ldots.$ in the boson case. The main problems of Majorana theory are the decreasing mass spectrum (1.5), absence of antiparticles and troublesome tachyonic solutions - the problems common to high spin theories. Nevertheless one can interpret the Majorana theory as a natural way to incorporate these *additional* degrees of freedom into the relativistic Dirac equation. Indeed we have found the solution of the Majorana commutation relations and the corresponding equation which exhibits an increasing mass spectrum (1.8) and nonzero string tension (1.6). [2]. Unlike Majorana we shall consider the *infinite sequence* of high-dimensional representations of the Lorentz group with nonzero Casimir operators $(\vec{a}\cdot\vec{b})$

[2]An alternative way to incorporate the internal motion into the Dirac equation was suggested by Pierre Ramond in 1971 [13] In his extension of the Dirac equation the internal motion is incorporated through the construction of operator-valued gamma matrices. The equations which define the Γ_{μ} matrices are

$$< \Gamma_{\mu}(\tau) >= \gamma_{\mu}\,,$$

$$\{\Gamma_{\mu}(\tau), \Gamma_{\nu}(\tau^{'})\} = 2\eta_{\mu\nu}\delta(\frac{1}{2\pi\alpha^{'}}(\tau - \tau^{'}))$$

$$\Gamma^{+}_{\mu}(\tau)\gamma_0 = \gamma_0\Gamma_{\mu}(\tau)\,,$$

where it is required that the proper-time average $< \ldots >$ over the periodic internal motion with period and $(\vec{a}^2 - \vec{b}^2)$. The important restriction to be imposed on the system is that it should have a point particle limit. In the given case this restriction should be understood as a principle according to which the infinite sequence of representations should contain the Dirac spin one-half representation. In the next section the known representations of the Lorentz group $SO(3,1)$ are reviewed and necessary representations are selected. These representations (j_0, λ) and their adjoint $(j_0, -\lambda)$ are enumerated by the index $r = j_0 + 1/2$, where $r = 1, \ldots, N$ and $j_0 = 1/2, 3/2, \ldots$ is the lower spin in the representation (j_0, λ), thus $j = j_0, j_0 + 1, \ldots$. [3]

We shall introduce also the concept of the *dual representation* which is defined as $\Theta = (j_0; \lambda) \to (\lambda; j_0) = \Theta^{dual}$. The important property of the dual representations Θ and Θ^{dual} is that for them the matrix elements of Lorentz generators $I_{\mu\nu}$ are *precisely the same*, the only difference between them is that the lower spin is equal to j_0 for the reperesentation Θ and is equal to λ for its dual Θ^{dual} (see formulas (2.9)(2.10) and (2.5)). Therefore any solution Γ_μ of the Majorana commutation relations (1.4) for Θ can be translated into the corresponding solution Γ^{dual}_μ for Θ^{dual}, but with the different spin contents and thus with different spectrum. This dual transformation is essentially used in subsequent sections. The reason is that it is much easier and transparent to solve Majorana commutation relations for small values of $N = 1, 2, \ldots,$ (at any stage we have to deal with finite size matrices and one can control the spectral properties of the problem) and then at the end to make the dual transformation.

The invariant Lagrangian, the string equation and the Majorana commutation relations are formulated in the second section. In the third section the solution of Majorana commutators is constructed for an arbitrary N in the form of Jacoby matrices [16]. This basic solution (B-solution) is well defined and the corresponding spectrum can be computed for any N, but the Γ_0 matrix is not Hermitian. In the forth section the necessary Hermitian property of the system and the correct commutation relation with the matrix of the invariant form have been found by changing the phases of the B-solution. The solution is symmetric Σ-solution and the spectrum of the theory now consists of particles and antiparticles of increasing half-integer spin and lying on quasilinear

$2\pi\alpha' = 1/\sigma$ should coincide with the Dirac matrices. The mass spectrum lies on linear trajectories

$$M_n^2 = \frac{1}{\alpha'}(j + n - 3/2)\,, \qquad n = 1, 2, 3, \ldots$$

where σ is the string tension, $j = 1/2, 3/2, \ldots$ and n enumerates the trajectories (here part of the states are spurious). In both cases one can see effectively an extensions of Dirac gamma matrices into the infinite-dimensional case, but these extensions are quiet different. The free Ramond string is a consistent theory in ten dimensions and the spectrum contains a massless ground state. For the subsequent development of superstring theories and their unification into a single M-theory see [15]. For our purposes we shall follow Majorana's approach to incorporate the internal motion in the form of an infinite-dimensional wave equation.

[3] The representations used in the Dirac equation are $(1/2, -3/2)$ and $(1/2, 3/2)$ and in the Majorana equation they are $(0, 1/2)$ in the boson case and $(1/2, 0)$ in the fermion case.

trajectories of different slope

$$2\pi\sigma_n = \frac{1}{\alpha_n^{'}} = \frac{2M^2}{n}, \qquad n = 1, 2, 3.... \tag{1.6}$$

here n enumerates the trajectories and the lower spin on a trajectory is $j = n + 1/2$. *This result demonstrates that we have indeed a string equation which has trajectories with different string tension and that trajectories with large n are almost "free" because the string tension tends to zero.* The number of particles with a given spin j is equal to $j + 1/2$. The unwanted property of the Σ-solution is that the smallest mass on a given trajectory n has the spin $j = n + 1/2$ and decreases as $3M^2/n(n + 3)$.

In the fifth section additional $(\vec{a} \cdot \vec{b})\Gamma_5$ and pure Casimir $(\vec{a}^2 - \vec{b}^2)$ mass terms is introduced into the string equation (1.3)

$$\{i\Gamma_\mu \partial^\mu - M(\vec{a} \cdot \vec{b})\Gamma_5 - gM(\vec{a}^2 - \vec{b}^2)\}\Psi = 0, \tag{1.7}$$

where $(\vec{a} \cdot \vec{b})$ and $(\vec{a}^2 - \vec{b}^2)$ are the Casimir operators of the Lorentz algebra. These terms essentially increase the string tension, so that all trajectories have a nonzero slope, but still we have a decreasing part in the spectrum.

The unwanted property of the Σ-solution, that is the decreasing of the smallest mass on a given trajectory, can be solved by dual transformation of the system. Indeed the dual transformation *does not affect the higher spin states and thus does not change the slope of the trajectories* (1.6), but has the commulative effect on lower spin states keeping them bounded from below. The dual formula for the mass spectrum is

$$M_n^2 = \frac{2M^2}{n} \frac{(j + n)(j + n + 1)}{j + (n + 1)/2}, \qquad n = 1, 2, \ldots \tag{1.8}$$

The lower spin on a given trajectory is either 1/2 or 3/2 depending on n: if n is odd then $j_{min} = 1/2$, if n is even then $j_{min} = 3/2$. *Thus the main problem of decreasing spectrum has been solved after the dual transformation because the spectrum is now bounded from below.* Additional Γ_5 mass term and the pure Casimir mass term is also analyzed. In the last, seventh, section the equation is extended to include bosons and have been shown that bosons also lie on the same quasilinear trajectories (1.8). At the end the problems of tachyonic and ghost solutions of the Majorana and of the new equation is discussed. This equation has its own value independent of the motivation advocated in this article. Some technical details can be found in the Appendixes.

2. Representations of Lorentz algebra. Majorana commutation relations

The algebra of the $SO(3,1)$ generators [18, 12]

$$[I^{\mu\nu} I^{\lambda\rho}] = -\eta^{\mu\lambda} I^{\nu\rho} + \eta^{\nu\lambda} I^{\mu\rho} + \eta^{\mu\rho} I^{\nu\lambda} - \eta^{\nu\rho} I^{\mu\lambda}$$

can be rewritten in terms of $SO(3)$ generators $\vec{a}$ and Lorentz boosts $\vec{b}$

$$
\begin{aligned}
a_x = iI^{23} \;\; a_y = iI^{31} \;\; a_z = iI^{12} \\
b_x = iI^{10} \;\; b_y = iI^{20} \;\; b_z = iI^{30}
\end{aligned}
$$

as [12] (we use Majorana's notations)

$$
[a_x, a_y] = ia_z \tag{2.1}
$$

$$
[a_x, b_y] = ib_z \tag{2.2}
$$

$$
[b_x, b_y] = -ia_z\,. \tag{2.3}
$$

The irreducible representations $R^{(j)}$ of the $SO(3)$ algebra (2.1) are

$$
\begin{aligned}
< j, m|a_z|j, m > &= m \\
< j, m|a_+|j, m-1 > &= \sqrt{(j+m)(j-m+1)} \\
< j, m|a_-|j, m+1 > &= \sqrt{(j+m+1)(j-m)}\,,
\end{aligned} \tag{2.4}
$$

where $m = -j, \ldots, +j$,the dimension of $R^{(j)}$ is $2j+1$ and $j = 0, 1/2, 1, 3/2, \ldots$ Then

$$
< j, m|\vec{a}^2|j, m >= j(j+1)\,.
$$

Since $\vec{b}$ transforms as a vector under $SO(3)$ spatial rotations (2.2) it follows that

$$
< j|\vec{b}|j' >= 0
$$

unless $j' = j$, or $j' = j \pm 1$, thus the matrix elements of $\vec{b}$ are fixed by their vector character, apart from the factors depending on j but not on m. Therefore the solution of the commutation relations (2.2) for $\vec{b}$ can be parameterized as [18, 12]

$$
\begin{aligned}
< j, m|b_z|j, m > &= \lambda_j \cdot m \\
< j-1, m|b_z|j, m > &= \varsigma_j \cdot \sqrt{(j^2 - m^2)} \\
< j, m|b_z|j-1, m > &= \varsigma_j \cdot \sqrt{(j^2 - m^2)}\,,
\end{aligned} \tag{2.5}
$$

plus similar formulas for the b_x and b_y generators. The amplitudes λ_j describe *diagonal* transitions inside the $SO(3)$ multiplet $R^{(j)}$, while ς_j describe *nondiagonal* transitions between $SO(3)$ multiplets which form the representation Θ of $SO(3,1)$. These amplitudes should satisfy the boundary condition

$$
(j_0 + 1)\lambda_{j_0} = i\lambda \qquad \varsigma_{j_0} = 0\,, \tag{2.6}
$$

where j_0 defines the lower spin in the representation $\Theta = (j_0; \lambda)$ and λ is a free parameter, thus

$$
\Theta(j_0, \lambda) = \oplus \sum_{j=j_0}^{\infty} R^{(j)}\,. \tag{2.7}
$$

The amplitudes λ and ς can be found from the last of the commutation relations (2.3) rewritten in the component form

$$\begin{aligned} &(2j+3)\cdot\varsigma_{j+1}^2 - (2j-1)\cdot\varsigma_j^2 - \lambda_j^2 = 1 \\ &\{(j+1)\cdot\lambda_j - (j-1)\cdot\lambda_{j-1}\}\cdot\varsigma_j = 0\,. \end{aligned} \tag{2.8}$$

From the last recursion equation and the boundary conditions (2.6) it follows that

$$\lambda_j = \frac{j-1}{j+1}\lambda_{j-1} = \frac{j_0(j_0+1)}{j(j+1)}\lambda_{j_0} = i\frac{j_0\lambda}{j(j+1)}\,, \tag{2.9}$$

where j_0 is the lower spin in the representation $\Theta = (j_0;\lambda)$ and λ appears as an essential dynamical parameter which cannot be determined solely from the kinematics of the Lorentz group. Substituting λ_j from (2.9) into (2.8) one can find ς_j

$$\varsigma_j^2 = \frac{(j^2-j_0^2)(j^2-\lambda^2)}{j^2(4j^2-1)} \tag{2.10}$$

in terms of j_0 and λ. [4] The adjoint representation is defined as $\dot\Theta = (j_0;-\lambda)$.

We shall introduce here a new concept of the *dual representation* which is defines as follow

$$\Theta = (j_0;\lambda) \longleftrightarrow (\lambda;j_0) = \Theta^{dual}\,. \tag{2.11}$$

From formulas (2.9),(2.10) and (2.5) it is easy to see that representations (j_0,λ) and $(-j_0,-\lambda)$ should be considered as identical. Therefore the dual transformation of the adjoint representation $\dot\Theta = (j_0;-\lambda)$ which is defined as $(-\lambda;j_0)$ is identical with $(\lambda;-j_0)$, thus

$$\dot\Theta = (j_0;-\lambda) \longleftrightarrow (\lambda;-j_0) = \dot\Theta^{dual}\,. \tag{2.12}$$

The important property of the dual representations Θ and Θ^{dual} is that the matrix elements of the Lorentz generators $I_{\mu\nu}$, which are defined by the formulas (2.9), (2.10) and (2.5), are *precisely the same.* The only difference between them is that the lower spin is equal to j_0 for the reperesentation Θ and is equal to λ for its dual Θ^{dual}. This property of dual representations allows to transform any solution Γ_μ of Majorana commutation relations into its dual ones Γ_μ^{dual}, simply exchanging j_0 for λ.

Let us consider pairs of adjoint representations with $j_0 = 1/2, 3/2, \cdots$

$$\Theta_r = (j_0;\lambda) \qquad \Theta_{\dot r} = (j_0;-\lambda) \tag{2.13}$$

[4] The representations used in the Dirac equation are $(1/2,-3/2)$ and $(1/2,3/2)$ and in the Majorana equation they are $(0,1/2)$ in the boson case and $(1/2,0)$ in the fermion case. The infinite-dimensional Majorana representation $(1/2,0)$ contains $j = 1/2, 3/2, \ldots$ multiplets of the $SO(3)$ while $(0,1/2)$ contains $j = 0,1,2,\ldots$ multiplets. The essential difference between these representations is that the Lorentz boost operators $\vec b$ are diagonal in the first case (the diagonal amplitudes (2.9) are $\lambda^1_{1/2} = -\lambda^{\dot 1}_{1/2} = i$ and nondiagonal amplitudes (2.10) are equal to zero $\varsigma^1_{3/2} = \varsigma^{\dot 1}_{3/2} = 0$) and they are nondiagonal in the case of Majorana representations (the nondiagonal amplitudes are $\varsigma_j = 1/2$ and the diagonal amplitudes are equal to zero $\lambda_j = 0$ as it follows from (2.9) and (2.10) and coincide with the solution (19) in [12]).

which we shall enumerate by the representation index $r = 1, 2, 3, \ldots$, so that $j_0 = r - 1/2$ (see fig. 1). The corresponding matrices $\vec{b}^r$ and $\vec{b}^{\dot r}$ are defined by (2.5) where from (2.9) and (2.10) we have

$$\lambda_j^r = -\lambda_j^{\dot r} = i\frac{r - 1/2}{j(j+1)}\lambda \qquad j \geq r - 1/2 \tag{2.14}$$

and

$$\varsigma_j^r = \varsigma_j^{\dot r} = \frac{1}{2}\sqrt{(1 - \frac{r^2 - r}{j^2 - 1/4})(1 - (\frac{\lambda}{j})^2)} \qquad j \geq r + 1/2 \tag{2.15}$$

and we shall consider the case when

$$-3/2 \leq \lambda \leq 3/2 \tag{2.16}$$

to have ς real for all values of r. The Casimir operators $(\vec{a} \cdot \vec{b})$ and $(\vec{a}^2 - \vec{b}^2)$ for the representation Θ_r are equal correspondingly to

$$< j, m|\vec{a} \cdot \vec{b}|j, m > = i\lambda(r - 1/2) \tag{2.17}$$

$$< j, m|(\vec{a}^2 - \vec{b}^2)|j, m > = (r - 1/2)^2 + \lambda^2 - 1\,. \tag{2.18}$$

As it is easy to see from these formulas the Casimir operator $(\vec{a} \cdot \vec{b})$ is nonzero only if $\lambda \neq 0$. For the Majorana representations $(1/2, 0)$ and $(0, 1/2)$ the Casimir operator $(\vec{a} \cdot \vec{b})$ is equal to zero.

To have an invariant scalar product

$$< \Theta\Psi_1|\Theta\Psi_2 > = < \Psi_1|\Psi_2 >, \tag{2.19}$$

where $\Theta = 1 + \frac{1}{2}\epsilon_{\mu\nu}I^{\mu\nu}$, one should define the matrix Ω

$$< \Psi_1|\Psi_2 > = \bar{\Psi}_1\Psi_2 = \Psi_1^+\Omega\Psi_2 = \Psi^{*r}_{1jm}\Omega^{rr'}_{jmj'm'}\Psi^{r'}_{2j'm'}\,, \tag{2.20}$$

with the properties

$$\Omega a_k = a_k\Omega \tag{2.21}$$

$$\Omega b_k = b_k^+\Omega \tag{2.22}$$

$$\Omega = \Omega^+\,. \tag{2.23}$$

From the first relation it follows that

$$\Omega = \omega_j^{rr'} \cdot \delta_{jj'} \cdot \delta_{mm'} \tag{2.24}$$

and from the last two equations, for our choice of the representation Θ (2.13)-(2.16) and for a real λ,

$$\omega_j^{r\dot r} = \omega_j^{\dot r r} = 1\omega_j^2 = 1\,, \tag{2.25}$$

thus Ω is an antidiagonal matrix

$$\bar{\Psi}_1\Psi_2 = \sum_{r=1}^{\infty}\ \sum_{j=r-1/2}^{\infty}\ \sum_{m=-j}^{j} (\Psi^{*r}_{1jm}\Psi^{\dot r}_{2jm} + \Psi^{*\dot r}_{1jm}\Psi^{r}_{2jm}) \tag{2.26}$$

and to have a finite invariant product the summation should be convergent.

Having in hand the invariant form (2.26) one can construct the Lagrangian

$$L = \int \{ i\bar{\Psi}\Gamma_\mu \partial^\mu \Psi - i\partial^\mu \bar{\Psi}\Omega\Gamma_\mu^+ \Omega\Psi - 2M\bar{\Psi}\Psi \} d^4x \,, \tag{2.27}$$

the corresponding equation of motion

$$\{ i(\Gamma_\mu + \Omega\Gamma_\mu^+\Omega)\partial^\mu - 2M \}\Psi = 0 \tag{2.28}$$

and the current

$$J_\mu = i\bar{\Psi}\frac{(\Gamma_\mu + \Omega\Gamma_\mu^+\Omega)}{2}\Psi \qquad \partial^\mu J_\mu = 0 \,. \tag{2.29}$$

The crucial point is that current density J_0 should be positive definite, which is equivalent to the positivity of the eigenvalues of the matrix $\rho = (\Omega\Gamma_0 + \Gamma_0^+\Omega)/2$. If in addition the important relation is required to be satisfied

$$\Gamma_\mu^+\Omega = \Omega\Gamma_\mu \,, \tag{2.30}$$

then the current will have the form $J_\mu = i\bar{\Psi}\Gamma_\mu\Psi$, and equation (2.28) can be transformed in this case into Hamiltonian form with the Hamiltonian $H = \Gamma_0^{-1}\vec{\Gamma}\vec{p} + \Gamma_0^{-1}M$.

The Lagrangian and the equation

$$\{ i\Gamma_\mu \partial^\mu - M \}\Psi = 0 \tag{2.31}$$

should be invariant under Lorentz transformations

$$X'_\mu = \Lambda_\mu^\nu X_\nu \qquad \Psi'(X') = \Theta(\Lambda)\Psi(X) \,, \tag{2.32}$$

which leads to the following equation for the gamma matrices

$$\Gamma_\nu = \Lambda_\nu^\mu \Theta \Gamma_\mu \Theta^{-1} \,. \tag{2.33}$$

If we use the infinitesimal form of Lorentz transformations

$$\Lambda_{\mu\nu} = \eta_{\mu\nu} + \epsilon_{\mu\nu} \qquad \Theta = 1 + \frac{1}{2}\epsilon_{\mu\nu} I^{\mu\nu} \tag{2.34}$$

it follows that gamma matrices should satisfy the Majorana commutation relation [12]

$$[\Gamma_\mu, I_{\lambda\rho}] = \eta_{\mu\lambda}\Gamma_\rho - \eta_{\mu\rho}\Gamma_\lambda \tag{2.35}$$

or in components (see formulas (13) in [12])

$$[\Gamma_0, a_x] = 0 \qquad [\Gamma_0, b_x] = i\Gamma_x \tag{2.36}$$

$$[\Gamma_x, a_x] = 0 \qquad [\Gamma_x, a_y] = i\Gamma_z \qquad [\Gamma_x, a_z] = -i\Gamma_y \tag{2.37}$$

$$[\Gamma_x, b_x] = i\Gamma_0 \qquad [\Gamma_x, b_y] = 0 \qquad [\Gamma_x, b_z] = 0 \,. \tag{2.38}$$

From these equations it follows that Γ_μ should satisfy the equation[5]

$$[[\Gamma_\mu, b_k], b_k] = -\Gamma_\mu \,, \qquad \mu = 0, k = x, y, z; \qquad \mu = k = x, y, z \tag{2.39}$$

and that $\Gamma_k = i[b_k, \Gamma_0]$,$k = x, y, z$. Therefore is it sufficient to solve equations (2.39) only for Γ_0. There are three equations which contain Γ_0 in (2.39) and one can see that only one of them is independent (the other two follow from the given one). One can also derive that

$$\{\Gamma_0 \Gamma_k\} = i[b_k \Gamma_0^2] \,, \tag{2.40}$$

thus the anticommutator between Γ_0 and Γ_k essentially depends on the form of the Γ_0^2 operator. These are the most important equations because they allow to find gamma matrices when a representation Θ of the Lorentz algebra is given. Let us choose N pairs of adjoint representations with $j_0 = 1/2, \ldots, N - 1/2$ (see fig. 1).

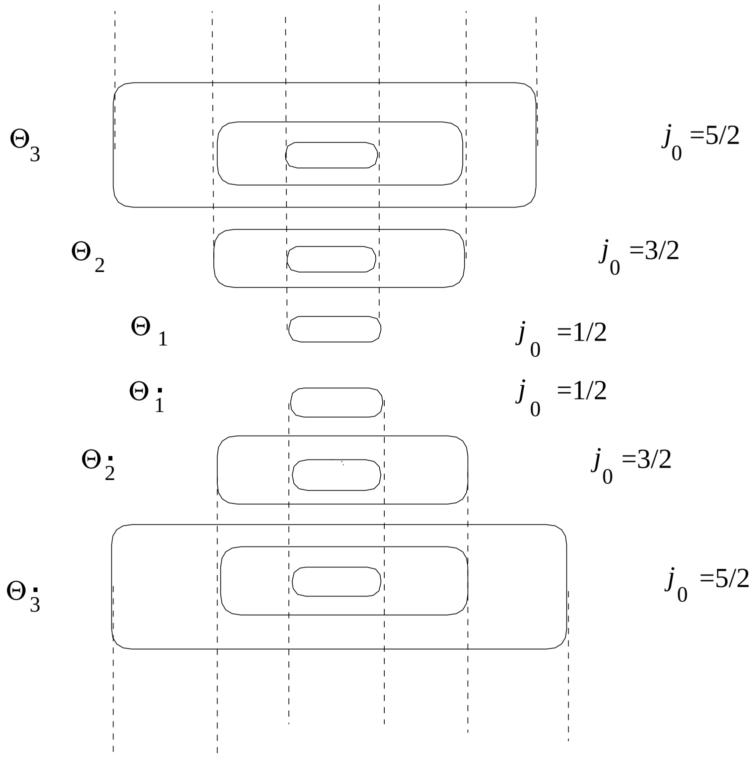

Figure 1: The tower of infinite-dimensional representations of Lorentz group $\Theta = (\Theta_{\dot N}, \cdots, \Theta_{\dot 1}, \Theta_1, \cdots, \Theta_N)$. The number of states on the spin level j is equal to j+1/2.

Because Γ_0 commutes with spatial rotations $\vec{a}$ it should have the form

$$< j, m|\Gamma_0^{rr'}|j'm' >= \gamma_j^{rr'} \cdot \delta_{jj'} \cdot \delta_{mm'} \,, \qquad r, r' = \dot N, \ldots, \dot 1, 1, \ldots, N \tag{2.41}$$

where we consider N pairs of adjoint representations $\Theta = (\Theta_{\dot N}, \cdots, \Theta_{\dot 1}, \Theta_1, \cdots, \Theta_N)$, thus $\gamma^{rr'}$ is $2N \times 2N$ matrix which should satisfy the equation (2.39) for $\mu = 0$ and

[5]If $\Gamma_\mu^{(1)}$ and $\Gamma_\mu^{(2)}$ are two solutions of the equation (2.39), then the sum is also a solution and if Γ_μ is a solution then using (2.22) one can see that $\Omega\Gamma_\mu^+\Omega$ is also a solution.

$k = z$

$$\Gamma_0 b_z^2 - 2 b_z \Gamma_0 b_z + b_z^2 \Gamma_0 = -\Gamma_0 \,. \tag{2.42}$$

Let us search the solutions of the above equation in the form of Jacoby matrices [16] (see also Appendix A)

$$\gamma_j = \begin{pmatrix} 0, & \gamma_j^{NN-1}, & & & \\ \gamma_j^{N-1N}, & 0, & \gamma_j^{N-1N-2}, & & \\ & \ddots & \ddots & \ddots & \\ & & \gamma_j^{\dot{N-1}\dot{N-2}}, & 0, & \gamma_j^{\dot{N-1}\dot{N}} \\ & & & \gamma_j^{\dot{N}\dot{N-1}}, & 0 \end{pmatrix} . \tag{2.43}$$

It should be understood that $\vec{a}^{rr'} = \delta^{rr'} \cdot \vec{a}$ and $\vec{b}^{rr'} = \delta^{rr'} \cdot \vec{b}^r$. The wave function has the form (see fig. 1)

$$\begin{pmatrix} \Psi^r_{jm} \\ \Psi^{\dot{r}}_{jm} \end{pmatrix} \qquad r = 1, \ldots, N \qquad j = r - 1/2, r + 1/2, \ldots \qquad m = -j, \ldots, j. \tag{2.44}$$

The equation (2.42) together with the matrix elements (2.41),(2.43) and (2.5), (2.9),(2.10) completely define the problem. It is important that any solution Γ_0 of the above equation for the representation Θ can be translated into its dual solution Γ_0^{dual} simply by exchanging j_0 for λ and with the spin j runing in a different interval. This follows from the fact that the matrix elements of the Lorentz generators for the dual representations are precisely the same.

The solutions of the equation (2.42) are defined up to a set of constant factors which are independent from j (see Appendix A). Indeed, because Jacoby matrices have very specific form, the original equations factorize into separate equations for every element γ_j^{rr+1} of the Jacoby matrix (2.43) and one can check that the solution has the form (see Appendix A)

$$\gamma_j^{rr+1} = Const \sqrt{(1 - \frac{r^2}{N^2})} \cdot \sqrt{(\frac{j^2 + j}{4r^2 - 1} - \frac{1}{4})}$$

and has therefore $(4N - 2)$ *j-independent free constant.* We essentially use this freedom to impose necessary physical constraints on a solutions requiring correct behaviour of the spectrum, Hermitian property of Γ_0, commutation relation with the matrix of invariant form and the positivity of the current density matrix. In general these are very complicated problems simply because one should study the spectral properties of the matrices of infinite size. Analysing the situation for some lower values of N [17] one can find solution with the required properties for arbitrary N [17]. The parameter N also play the role of the natural regularization.

The first inspection of this solution simply shows that every element γ_j^{rr+1} grows like $\approx j$ and in general all eigenvalues will also grow with j. Therefore the mass spectrum will have Majorana-like behaviour (1.5) $M_j \approx M/j$. To avoid this general behaviour of the spectrum one should carefully inspect eigenvalues of the matrix Γ_0 for small values of N and then for arbitrary N. The solution which appear has exeptional behaviour: half of the eigenvalues of the spectrum are increasing and the other half are decreasing. One can achieve this exeptional behaviour by tuning the free constants in the general solution. One should take into account that matrix elements γ_j^{rr+1} have very comlicated "root" dependence on j and this fact makes complite investigation of the spectrum extremely complicated, but one can manage to solve the problem exactly [17].

3. N pairs of infinite-dimensional representations

Let us take N pairs of adjoint representations $\Theta = (\Theta_{\dot N}, \ldots, \Theta_{\dot 1}, \Theta_1, \ldots, \Theta_N)$ which can be rewritten also in the form (2.7)

$$(N-1/2; -\lambda), \ldots, (1/2; -\lambda), (1/2; \lambda), \ldots, (N-1/2; \lambda)\,, \tag{3.1}$$

the wave function has the form

$$\Psi = \begin{pmatrix} \Psi^{r}_{jm} \\ \Psi^{\dot r}_{jm} \end{pmatrix} \qquad r = 1, \ldots, N; \qquad j = r - 1/2, r + 1/2, \ldots; \qquad m = -j, \ldots, j$$

and the transition amplitudes are given in (2.14) and (2.15). The basic solution (*B-solution*) of the equation (2.42) for the Γ_0 in this general case is a $2N \times 2N$ Jacoby matrix [17]. The B-solution has the following nonzero elements (see also Appendix A)

$$\gamma_j^{r+1 r} = \gamma_j^{r r+1} = \gamma_j^{\dot{r+1}\dot r} = \gamma_j^{\dot r \dot{r+1}} = i\sqrt{(1 - \frac{r^2}{N^2})(\frac{j^2+j}{4r^2-1} - \frac{1}{4})} \qquad j \geq r + 1/2 \tag{3.2}$$

$$\gamma_j^{1\dot 1} = \gamma_j^{\dot 1 1} = j + 1/2 \tag{3.3}$$

here $r = 1, \ldots, N-1$. The gamma matrices for the first few values of j are equal to

$$\gamma_{1/2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_{3/2} = \begin{pmatrix} 0 & i\sqrt{1-1/N^2} & & \\ i\sqrt{1-1/N^2} & 0 & 2 & \\ & 2 & 0 & i\sqrt{1-1/N^2} \\ & & i\sqrt{1-1/N^2} & 0 \end{pmatrix}, \quad \ldots$$

and they grow in size with j until $j = N - 1/2$, for greater j the size of the matrix γ_j remains the same and is equal to $2N \times 2N$. The determinant of the γ_j matrix for j in the interval $[3/2, 7/2, \ldots, N-1/2]$ is equal to

$$Det\gamma_j = (1 - 1/N^2)^2 \cdots (1 - (j-1/2)^2/N^2)^2 \tag{3.4}$$

and for j in the interval $[1/2, 5/2, \ldots, N-1/2]$ is

$$Det\gamma_j = (-1)(1-4/N^2)^2 \cdots (1-(j-1/2)^2/N^2)^2 \,. \tag{3.5}$$

From the determinant and the trace of the gamma matrix it follows that

$$\epsilon_1^2 \cdot \ldots \cdot \epsilon_{j+1/2}^2 = |Det\gamma_j| \qquad \epsilon_1^2 + \ldots + \epsilon_{j+1/2}^2 = \frac{1}{2} Tr\gamma_j^2 \qquad j \geq (N-1/2)\,. \tag{3.6}$$

The characteristic equations for these matrices are:

$$\begin{aligned}
\gamma_{1/2}^2 - 1 &= 0, \\
((\gamma_{3/2}-1)^2 - 1/N^2)((\gamma_{3/2}+1)^2 - 1/N^2) &= 0, \\
(\gamma_{5/2}^2 - 1)((\gamma_{5/2}-1)^2 - 4/N^2)((\gamma_{5/2}+1)^2 - 4/N^2) &= 0, \\
&\cdots\cdots\cdots \\
\cdots(\gamma_j^2 - (1+(j-5/2)/N)^2)(\gamma_j^2 - (1+(j-1/2)/N)^2) &= 0\,,
\end{aligned} \tag{3.7}$$

where in the last equation $j \geq N-1/2$.

The positive eigenvalues ϵ_j can now be found

$$\begin{array}{ccccccccccc}
 & & & & 1 & & & & & & j=1/2 \\
 & & & 1-1/N & & 1+1/N & & & & & j=3/2 \\
 & & 1-2/N & & 1 & & 1+2/N & & & & j=5/2 \\
 & 1-3/N & & 1-1/N & & 1+1/N & & 1+3/N & & & j=7/2 \\
1-4/N & & 1-2/N & & 1 & & 1+2/N & & 1+4/N & & j=9/2 \\
 & & & & \cdots & & & & & & \\
 & & & & \cdots & 1+(j-5/2)/N & & 1+(j-1/2)/N & & & j \geq N-1/2\,.
\end{array} \tag{3.8}$$

In fig. 2 one can see the spectrum of the matrix γ_j as a function of j.

The number of states with angular momentum j grows as $j+1/2$ and this takes place up to spin $j = N-1/2$. For the higher spins $j \geq N-1/2$ the number of states remains constant and is equal to N (see fig. 2). Thus the coefficient of proportionality drops N times compared with the Majorana solution (see Appendix A) and now *many eigenvalues are less than unity. This actually means that by increasing the number of representations one can slow down the growth of the eigenvalues.* The mass spectrum is bounded from below only if all eigenvalues are less than unity.

In the limit $N \to \infty$ the B-solution (3.2) is being reduced to the form

$$\gamma_j^{r+1\, r} = \gamma_j^{r\, r+1} = \gamma_j^{\dot{r+1}\, \dot{r}} = \gamma_j^{\dot{r}\, \dot{r+1}} = i\sqrt{\left(\frac{j^2+j}{4r^2-1} - \frac{1}{4}\right)} \qquad j \geq r+1/2 \tag{3.9}$$

$$\gamma_j^{1\dot{1}} = \gamma_j^{\dot{1}1} = j+1/2\,, \tag{3.10}$$

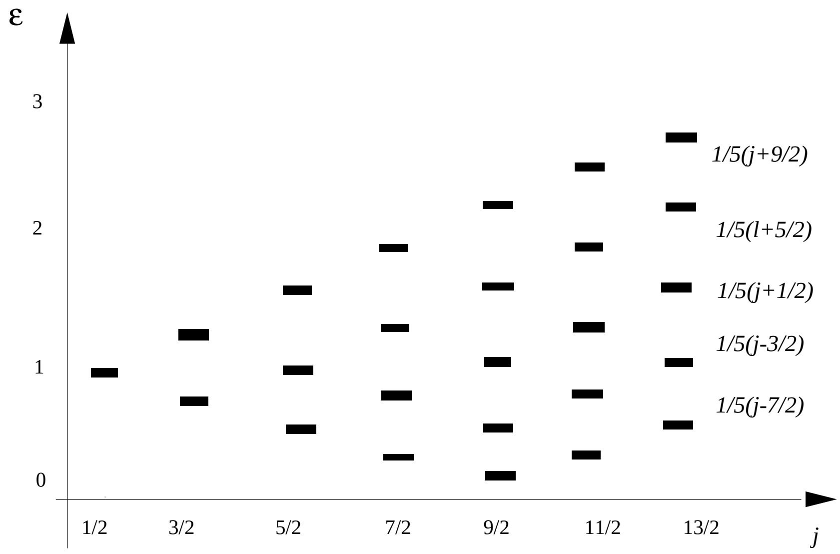

Figure 2: The positive part of spectrum of the Γ_0 matrix for N=5. The number of states with $\epsilon < 1$ increases. As N tends to infinity the spectral cone $j = [1/2, N - 1/2]$ becomes more "narrow" and the eigenvalues are concentrated around $\epsilon = 1$.

where $r = 1, 2, \ldots$. As it is easy to see from the previous formulas, all eigenvalues ϵ_j tend to unity when the number of representations $N \to \infty$. The characteristic equation which is satisfied by the gamma matrix in this limit is

$$(\gamma_j^2 - 1)^{j+1/2} = 0 \qquad j = 1/2, 3/2, 5/2, \ldots, \tag{3.11}$$

with all eigenvalues $\epsilon_j = \pm 1$.

The determinant and the trace are equal to

$$Det\gamma_j = \pm 1 \qquad Tr\gamma_j^2 = 2j + 1\,, \tag{3.12}$$

thus

$$\epsilon_1^2 \cdot \ldots \cdot \epsilon_{j+1/2}^2 = 1 \qquad \epsilon_1^2 + \ldots + \epsilon_{j+1/2}^2 = j + 1/2\,. \tag{3.13}$$

The matrix $\Omega\Gamma_0$ has the characteristic equation

$$(\omega_j\gamma_j - 1)^{2j+1} = 0 \tag{3.14}$$

with all eigenvalues equal to $\rho_j = +1$, therefore introducing $\omega_j\gamma_j - 1 = \sigma_j$ we have an important relation $\sigma_j^{2j+1} = 0$ and $\gamma_j = \omega_j(1 + \sigma_j)$. The σ_j defines the algebra of elements

$$\sigma_j, \sigma_j^+, \sigma_j^2, \sigma_j^{+2}, \ldots, \sigma_j^{2j}, \sigma_j^{+2j}\,, \tag{3.15}$$

with the property

$$\sigma_j^{2j+1} = \sigma_j^{+2j+1} = 0\,. \tag{3.16}$$

Thus the matrix $\Omega\Gamma_0$ is positive definite and all its eigenvalues are equal to one, but the important relations

$$\Omega\Gamma_0 \neq \Gamma_0^+\Omega \qquad \Gamma_0^+ \neq \Gamma_0 \tag{3.17}$$

do not hold and therefore the Hamiltonian is not Hermitian. What is crucial here is that we can continue to improve the B-solution without disturbing its determinant which is equal to one (3.12). The last property of the determinat is necessary in order to have spectrum which is symmertically distributed around unity. In the next section we shall find the Hermitian solution for Γ_0 using the fact that one can change the phases of the matrix elements of the B-solution without disturbing its determinant.

4. Hermitian solution for Γ_0

The Hermitian solution (*H-solution*) of (2.42) for Γ_0 can be found as a phase modification of the basic *B-solution* (3.2)

$$\gamma_j^{r+1r} = -\gamma_j^{rr+1} = -\gamma_j^{\dot{r+1}\dot{r}} = \gamma_j^{\dot{r}\dot{r+1}} = i\sqrt{(\frac{j^2+j}{4r^2-1} - \frac{1}{4})} \qquad j \geq r+1/2 \tag{4.1}$$

$$\gamma_j^{1\dot{1}} = \gamma_j^{\dot{1}1} = j + 1/2\,. \tag{4.2}$$

These matrices are Hermitian $\Gamma_0^+ = \Gamma_0$, but the characteristic equations are more complicated now:

$$\begin{aligned}
\gamma_{1/2}^2 - 1 &= 0,\\
(\gamma_{3/2}^2 - 2\gamma_{3/2} - 1)(\gamma_{3/2}^2 + 2\gamma_{3/2} - 1) &= 0,\\
(\gamma_{5/2}^2 - 1)(\gamma_{5/2}^2 - 4\gamma_{5/2} + 1)(\gamma_{5/2}^2 + 4\gamma_{5/2} + 1) &= 0,\\
(\gamma_{7/2}^4 + 4\gamma_{7/2}^3 - 6\gamma_{7/2}^2 - 4\gamma_{7/2} + 1)(\gamma_{7/2}^4 - 4\gamma_{7/2}^3 - 6\gamma_{7/2}^2 + 4\gamma_{7/2} + 1) &= 0,\\
(\gamma_{9/2}^2 - 1)(\gamma_{9/2}^4 + 4\gamma_{9/2}^3 - 14\gamma_{9/2}^2 + 4\gamma_{9/2} + 1)\cdot &\\
\cdot(\gamma_{9/2}^4 - 4\gamma_{9/2}^3 - 14\gamma_{9/2}^2 - 4\gamma_{9/2} + 1) &= 0,\\
\cdots &
\end{aligned} \tag{4.3}$$

These polynomials $p(\epsilon)$ have reflective symmetry and are even

$$p_j(\epsilon) = \epsilon^{2j+1} p_j(1/\epsilon) \tag{4.4}$$

$$p_j(-\epsilon) = p_j(\epsilon) \tag{4.5}$$

therefore if ϵ_j is a solution then $1/\epsilon_j$,$-\epsilon_j$ and $-1/\epsilon_j$ are also solutions. Computing the traces and determinants of these matrices one can get the following general relation for the eigenvalues

$$\epsilon_1^2 \cdot \ldots \cdot \epsilon_{j+1/2}^2 = 1 \qquad \epsilon_1^2 + \ldots + \epsilon_{j+1/2}^2 = j(2j+1)\,. \tag{4.6}$$

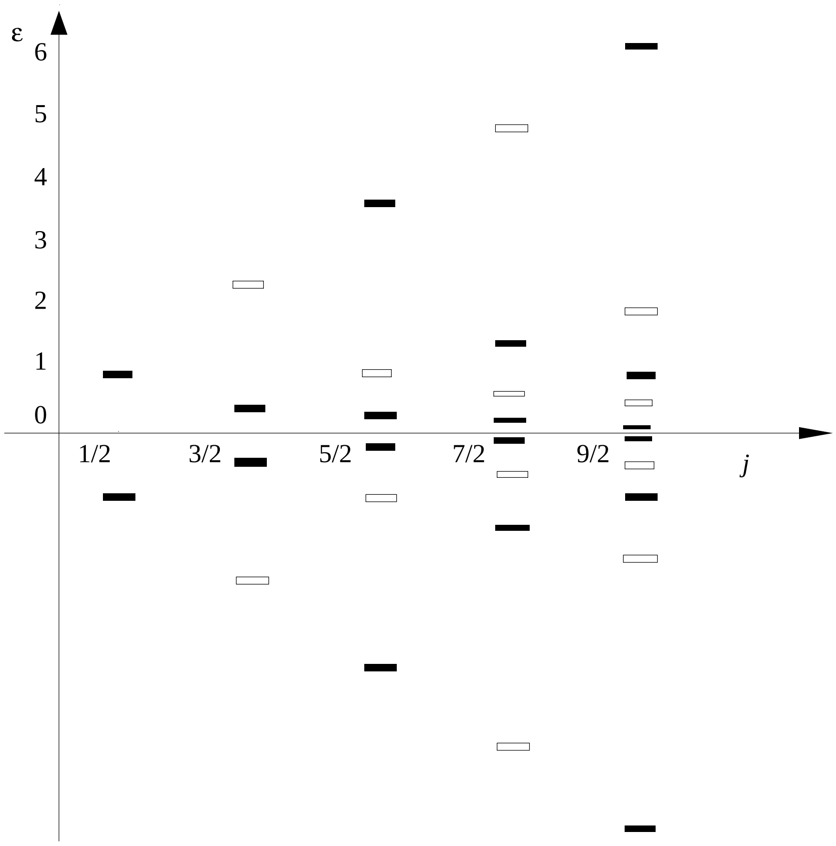

Figure 3: The spectrum of the Γ_0 matrix for the Hermitian solution (H-solution). The quasilinear trajectories have different slope. One can observe also approximate hyperbolic curves on which lie the eigenvalues from different trajectories. The symmetric solution (Σ-solution) has the same spectrum.

The eigenvalues ϵ_j can now be found

$$\begin{array}{rcll}
 & 1 & & j=1/2 \\
\sqrt{2}-1 & & \sqrt{2}+1 & j=3/2 \\
2-\sqrt{3} & 1 & 2+\sqrt{3} & j=5/2 \\
\sqrt{(\sqrt{2}+1)^2+1}-(\sqrt{2}+1) & & \sqrt{(\sqrt{2}-1)^2+1}-(\sqrt{2}-1) & j=7/2 \\
\sqrt{(\sqrt{2}-1)^2+1}+(\sqrt{2}-1) & & \sqrt{(\sqrt{2}+1)^2+1}+(\sqrt{2}+1) & j=7/2 \\
\sqrt{5}+1-\sqrt{(\sqrt{5}+1)^2-1} & & \sqrt{5}-1-\sqrt{(\sqrt{5}-1)^2-1} & j=9/2 \\
 & 1 & & j=9/2 \\
\sqrt{5}-1+\sqrt{(\sqrt{5}-1)^2-1} & & \sqrt{5}+1+\sqrt{(\sqrt{5}+1)^2-1} & j=9/2 \\
 & \dots & &
\end{array} \tag{4.7}$$

The changes in the phases of the matrix elements (4.1) result in a different behaviour of eigenvalues (see fig. 3).

The matrix $\Omega\Gamma_0$ has again the characteristic equation

$$(\omega_j\gamma_j - 1)^{2j+1} = 0 \tag{4.8}$$

and all eigenvalues are equal to one. Thus again the matrix $\Omega\Gamma_0$ is positive definite because all eigenvalues are equal to one, but the important relation

$$\Omega\Gamma_0 \neq \Gamma_0^+\Omega \tag{4.9}$$

does not hold. Introducing $\omega_j\gamma_j - 1 = \tau_j$ we have the relation $\tau_j^{2j+1} = 0$ and $\gamma_j = \omega_j(1+\tau_j)$. The τ_j defines the algebra of elements

$$\tau_j, \tau_j^+, \tau_j^2, \tau_j^{+2}, \ldots, \tau_j^{2j}, \tau_j^{+2j}, \tag{4.10}$$

with the property

$$\tau_j^{2j+1} = \tau_j^{+2j+1} = 0\,. \tag{4.11}$$

The solution of (2.42) for Γ_0 with both properties $\Gamma_0^+ = \Gamma_0$ and $\Omega\Gamma_0 = \Gamma_0^+\Omega$ can be found by using the basic solutions (3.2) rewritten with arbitrary phases of the matrix elements and then by requiring that Γ_0 should be Hermitian $\Gamma_0^+ = \Gamma_0$ and should satisfy the relation $\Omega\Gamma_0 = \Gamma_0^+\Omega$. This solution, Σ-*solution*, is symmetric and has the form

$$\gamma_j^{r+1r} = \gamma_j^{rr+1} = \gamma_j^{r\dot{+}1\dot{r}} = \gamma_j^{\dot{r}r\dot{+}1} = \sqrt{\left(\frac{j^2+j}{4r^2-1} - \frac{1}{4}\right)} \qquad j \geq r + 1/2 \tag{4.12}$$

$$\gamma_j^{1\dot{1}} = \gamma_j^{\dot{1}1} = j + 1/2\,, \tag{4.13}$$

which for the low values of j is:

$$\gamma_{1/2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_{3/2} = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & 2 & \\ & 2 & 0 & 1 \\ & & 1 & 0 \end{pmatrix}, \tag{4.14}$$

$$\gamma_{5/2} = \begin{pmatrix} 0 & \sqrt{1/3} & & & & \\ \sqrt{1/3} & 0 & \sqrt{8/3} & & & \\ & \sqrt{8/3} & 0 & 3 & & \\ & & 3 & 0 & \sqrt{8/3} & \\ & & & \sqrt{8/3} & 0 & \sqrt{1/3} \\ & & & & \sqrt{1/3} & 0 \end{pmatrix}, \ldots \tag{4.15}$$

In this case the Hermitian matrix $\Gamma_0^+ = \Gamma_0$ has the desired property

$$\Gamma_0^+\Omega = \Omega\Gamma_0\,. \tag{4.16}$$

This means that the current density is equal to $\rho = \Omega\Gamma_0$. In addition, all of the gamma matrices now have this property (2.30)

$$\Gamma_k^+\Omega = \Omega\Gamma_k \qquad k = x, y, z \tag{4.17}$$

which follows from the definition of $\Gamma_k = i[b_k\Gamma_0]$ and equation (2.22) $\Omega b_k = b_k^+\Omega$.

The characteristic equations and the spectrum (4.7) are the same for the Hermitian H-solution and symmetric Σ-solution (see fig. 3), but the corresponding characteristic equations for the matrices ρ_j are different

$$\begin{aligned}(\rho_{1/2} - 1)^2 &= 0,\\ (\rho_{3/2}^2 - 2\rho_{3/2} - 1)^2 &= 0,\\ (\rho_{5/2} + 1)^2(\rho_{5/2}^2 - 4\rho_{5/2} + 1)^2 &= 0,\\ (\rho_{7/2}^4 - 4\rho_{7/2}^3 - 6\rho_{7/2}^2 + 4\rho_{7/2} + 1)^2 &= 0,\\ (\rho_{9/2} - 1)^2(\rho_{9/2}^4 - 4\rho_{9/2}^3 - 14\rho_{9/2}^2 - 4\rho_{9/2} + 1)^2 &= 0,\\ &\cdots\end{aligned} \tag{4.18}$$

and the eigenvalues of the density matrix are therefore equal to

$$\begin{array}{cccc} 1 & & 1 & j = 1/2\\ 1-\sqrt{2} & & \sqrt{2}+1 & j = 3/2\\ 2-\sqrt{3} & -1 & 2+\sqrt{3} & j = 5/2\\ (1+\sqrt{2})+\sqrt{(\sqrt{2}+1)^2+1} & & (1+\sqrt{2})-\sqrt{(\sqrt{2}+1)^2+1} & j = 7/2\\ (1-\sqrt{2})+\sqrt{(\sqrt{2}-1)^2+1} & & (1-\sqrt{2})-\sqrt{(\sqrt{2}-1)^2+1} & j = 7/2\\ 1+\sqrt{5}+\sqrt{(\sqrt{5}+1)^2-1} & & 1+\sqrt{5}-\sqrt{(\sqrt{5}+1)^2-1} & j = 9/2\\ & 1 & & j = 9/2\\ 1-\sqrt{5}+\sqrt{(\sqrt{5}-1)^2-1} & & 1-\sqrt{5}-\sqrt{(\sqrt{5}-1)^2-1} & j = 9/2\\ & \cdots & & \cdots \end{array} \tag{4.19}$$

Both states with $j = 1/2$ have positive norms, the $j = 3/2$ level has two positive and two negative norm states, the $j = 5/2$ has four positive and two negative norm states, the $j = 7/2$ has four positive and four negative norm states, the $j = 9/2$ have six positive and four negative norm states, and so on (see fig. 3). In fig. 4 one can see the mass spectrum of the Σ-equation. The positive norm physical states, marked by the black bricks, are lying on the quasilinear trajectories and the negative norm ghost states, marked by white bricks, are also lying on the quasilinear trajectories of different slope. Thus the equation has the increasing mass spectrum, but the smallest mass on a given trajectory still tends to zero and in addition there are infinitely many ghost states. In the next section the solution of the problem of decreasing behaviour of the smallest mass on a given trajectory will be solved, but before that let us remark that one can project out the unwanted ghost states by constructing the projection operator

$$Pr = \prod_j (1 + \frac{1}{\epsilon_j}\omega_j\gamma_j)\,, \tag{4.20}$$

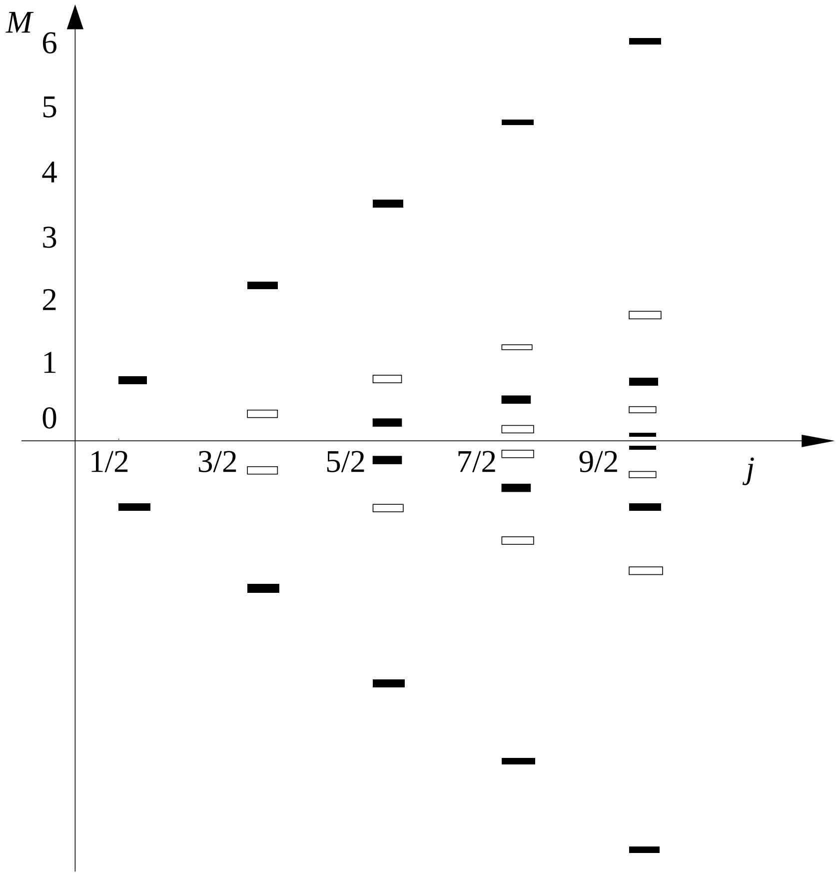

Figure 4: The mass spectrum of $H = M\Gamma_0^{-1}$. One can see positive norm physical states, marked by the black bricks, lying on the quasilinear trajectories and negative norm ghost states, marked by the white bricks, also lying on the quasilinear trajectories of different slope.

where the product is over ghost states (see also the end of this section).

In the case when some of the transition amplitudes in (4.12) are set to zero

$$\gamma^{\dot{1}1} = \gamma_j^{23} = \gamma_j^{45} = \ldots . = 0 \qquad \gamma^{1\dot{1}} = \gamma^{\dot{2}\dot{3}} = \gamma^{\dot{4}\dot{5}} = \ldots = 0 \tag{4.21}$$

and all other elements of the Γ_0 matrix remain the same as in (4.12) we have a new Σ_1-solution with the important property that Γ_0^2 is a diagonal matrix and that the antihermitian part of Γ_k anticommutes with Γ_0. Thus in this case we recover the nondiagonal part of the Dirac commutation relations for gamma matrices

$$\{\Gamma_0, \tilde{\Gamma_k}\} = 0 \qquad k = x, y, z. \tag{4.22}$$

For the solution (4.21) one can explicitly compute the slope of the trajectories. For the first one we have

$$M_1^2 = 2M^2(j-1) \approx 2M^2 j\,, \qquad j = 3/2, 7/2, \ldots \tag{4.23}$$

thus for this trajectory the string tension σ is equal to

$$2\pi\sigma_1 = \frac{1}{\alpha_1^{'}} = 2M^2\,. \tag{4.24}$$

For the second and third trajectories we have

$$M_2^2 = M^2\frac{(j^2-3j+2)}{(j-1/2)} \approx M^2 j\,, \qquad j = 5/2, 9/2, \ldots \tag{4.25}$$

$$M_3^2 = \frac{2M^2}{3}\frac{(j^2-5j+6)}{(j-1)} \approx \frac{2M^2}{3}j\,, \qquad j = 7/2, 11/2, \ldots \tag{4.26}$$

thus

$$2\pi\sigma_2 = \frac{1}{\alpha_2^{'}} = M^2, \qquad 2\pi\sigma_3 = \frac{1}{\alpha_3^{'}} = 2M^2/3\,, \tag{4.27}$$

and so on. When we move to the next trajectories the slope decreases and the string tension $\sigma_n = 1/2\pi\alpha_n^{'}$ varies from one trajectory to another and tends to zero

$$2\pi\sigma_n = \frac{1}{\alpha_n^{'}} = \frac{2M^2}{n} \to 0 \tag{4.28}$$

This result demonstrates that we have indeed the string equation which has trajectories with different string tension and that trajectories with large n are almost "free" because the string tension tends to zero. Finally, the general formula for all trajectories is

$$M_n^2 = \frac{2M^2}{n}\frac{j^2-(2n-1)j+n(n-1)}{j-(n-1)/2}\,, \qquad n = 1, 2, \ldots \tag{4.29}$$

where $j = n+1/2, n+5/2, \ldots$.(see fig. 5). The smallest mass on a given trajectory n has spin $j = n+1/2$ and decreases as

$$M_n^2(j = n+1/2) = \frac{3M^2}{n(n+3)}\,.$$

The other solution, Σ_2-solution, which shares the above properties of Σ_1-solution is (4.12) with

$$\gamma_j^{12} = \gamma_j^{34} = \ldots. = 0 \qquad \gamma_j^{\dot{1}\dot{2}} = \gamma_j^{\dot{3}\dot{4}} = \ldots = 0\,. \tag{4.30}$$

The difference between these last two solutions is that in the first case the lower spin is $j = 3/2$ and in the second case it is $j = 1/2$. The unwanted property of all these solutions Σ, Σ_1 and Σ_2 is that the smallest mass $M_n^2(min)$ tends to zero as it was in the original Majorana equation and the spectrum is not bounded from below. It is also true that $\Sigma = \Sigma_1 + \Sigma_2$ in the sense of the footnote [5].

We have to remark also that both equations, Σ_1 and Σ_2, which correspond to (4.21) and to (4.30) have natural *constraints*. In the rest frame they are

$$\begin{pmatrix}\psi_1\\ \psi_{\dot 1}\end{pmatrix}_{1/2} = \begin{pmatrix}\psi_3\\ \psi_{\dot 3}\end{pmatrix}_{5/2} = \cdots = \begin{pmatrix}\psi_{2k-1}\\ \psi_{\dot{2k}-1}\end{pmatrix}_{(4k-3)/2} = 0 \qquad k = 1, 2, \ldots \tag{4.31}$$

for the Σ_1 case (4.21) and

$$\begin{pmatrix}\psi_2\\ \psi_{\dot 2}\end{pmatrix}_{3/2} = \begin{pmatrix}\psi_4\\ \psi_{\dot 4}\end{pmatrix}_{7/2} = \cdots = \begin{pmatrix}\psi_{2k}\\ \psi_{\dot{2k}}\end{pmatrix}_{(4k-1)/2} = 0 \qquad k = 1, 2, \ldots \tag{4.32}$$

for the Σ_2 case (4.30).

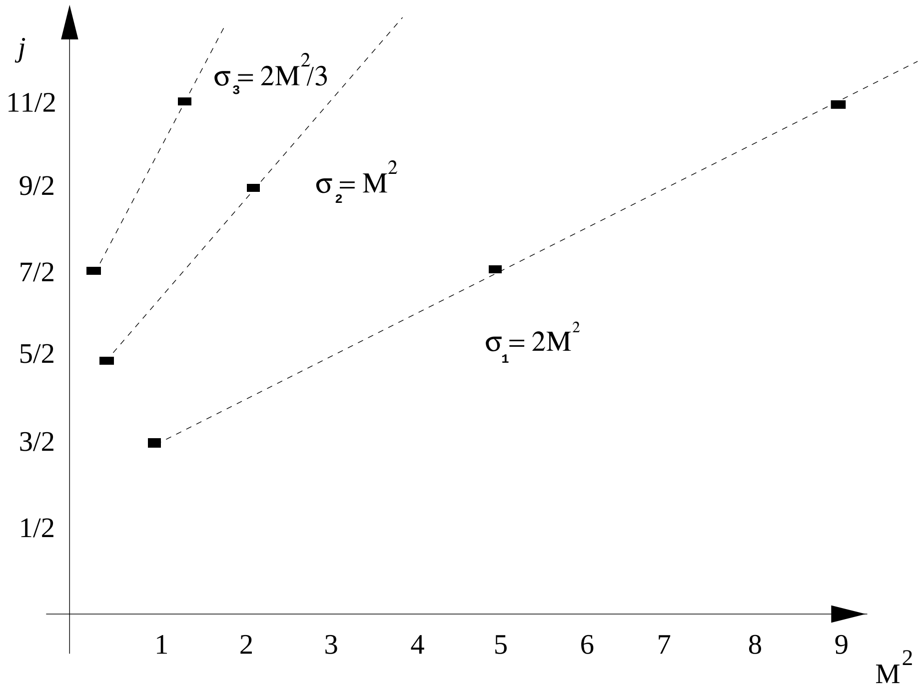

Figure 5: The mass spectrum has a structure of linear trajectories with different string tension $2\pi\sigma_n = 2M^2/n, n = 1, 2, ...$ The smallest mass on a given trajectory n has the spin $j = n + 1/2$ and tends to zero as $1/n^2$.

5. The Γ_5 mass term

One can define the Γ_5 matrix which has all the properties of Dirac's γ_5 matrix (see Appendix C) $\{\Gamma_5, \Gamma_\mu\} = 0$ and allows to introduce an additional mass term into the string equation (2.31) of the form

$$(\vec{a} \cdot \vec{b})\Gamma_5\,.$$

It was our aim to use those representations Θ of the Lorentz group which have nonzero Casimir operator

$$< j, m, r|\vec{a} \cdot \vec{b}|r, j, m > = i\lambda(r - 1/2)$$

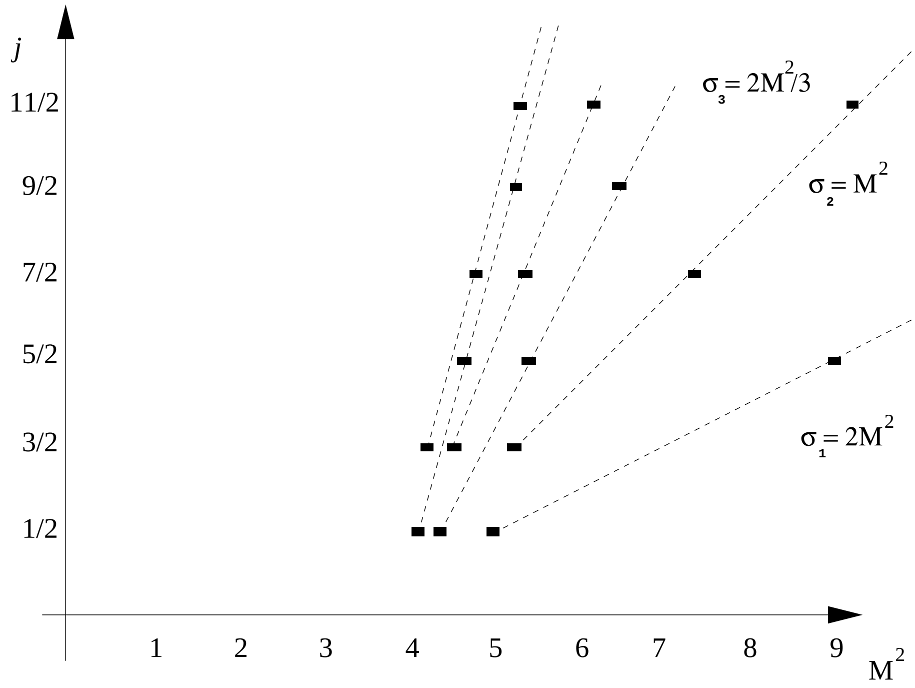

Figure 6: The spectrum of the dual equation has again a structure of linear trajectories with string tension $2\pi\sigma_n = 2M^2/n, n = 1, 2, ...$ The smallest mass on a given trajectory n has the spin $j = 1/2$ or $j = 3/2$ and is bounded from bellow.

$$< j, m, \dot{r}|\vec{a}\cdot\vec{b}|\dot{r}, j, m > = -i\lambda(r - 1/2)\,.$$

The matrix $(\vec{a}\cdot\vec{b})\Gamma_5$ is diagonal

$$\begin{aligned} < j, m, r|(\vec{a}\cdot\vec{b})\Gamma_5|r, j, m > &= < j, m, \dot{r}|(\vec{a}\cdot\vec{b})\Gamma_5|\dot{r}, j, m > \\ &= i\lambda M(-1)^{r+1}(r - 1/2)\,. \end{aligned}$$

With this new mass term the mass spectrum of the theory changes and we get trajectory

$$M_1^2 = 2M^2(j-1)\cdot\lambda^2 j(j+1) \approx 2(\lambda M)^2 j^3\,,$$

thus for this trajectory

$$2\pi\sigma_1 = \frac{1}{\alpha_1'} = 2(\lambda M)^2\cdot j^2\,.$$

For the second and third trajectories we have

$$M_2^2 = M^2\frac{(j^2-3j+2)}{(j-1/2)}\cdot\lambda^2(j-1)j \approx (\lambda M)^2 j^3\,,$$

$$M_3^2 = \frac{2M^2}{3}\frac{(j^2-5j+6)}{(j-1)}\cdot\lambda^2(j-2)(j-1) \approx \frac{2(\lambda M)^2}{3}j^3\,,$$

thus

$$2\pi\sigma_2 = \frac{1}{\alpha_2'} = (\lambda M)^2\cdot j^2 \qquad 2\pi\sigma_3 = \frac{1}{\alpha_3'} = 2(\lambda M)^2/3\cdot j^2\,, \tag{5.1}$$

and so on. The appearance of λ in these formulas together with M demonstrates that this contribution to the spectrum is possible only if we use representations which have nonzero Casimir operators. The general formula for all trajectories is

$$M_n^2 = \frac{2(\lambda M)^2}{n} \frac{j^2 - (2n-1)j + n(n-1)}{j - (n-1)/2} (j+1-n)(j+2-n) \qquad n = 1, 2, \ldots \quad (5.2)$$

and the smallest mass on a given trajectory behaves as

$$M_n^2(j = n+1/2) = \frac{3(\lambda M)^2}{n(n+3)} \frac{3}{2} \frac{5}{2} \qquad j = n + 1/2$$

and still falls to zero as $1/n^2$.

6. The dual string equation

Under the dual transformation (2.11)

$$\Theta = (j_0; \lambda) \to (\lambda; j_0) = \Theta^{dual}$$

the representation Θ (3.1) is transformed into its dual

$$\cdots\ (\lambda; -5/2)\ (\lambda; -3/2)\ (\lambda; -1/2)\ (\lambda; 1/2)\ (\lambda; 3/2)\ (\lambda; 5/2)\ \cdots \quad (6.1)$$

and we shall consider here the case $\lambda = 1/2$ in order to have the Dirac representation incorporated in Θ. The solution which is dual to Σ_2 (4.12) and (4.30) is equal to

$$\gamma_j^{r+1 r} = \gamma_j^{r r+1} = \gamma_j^{\dot{r+1} \dot{r}} = \gamma_j^{\dot{r} \dot{r+1}} = \sqrt{\left(\frac{1}{4} - \frac{j^2 + j}{4r^2 - 1}\right)}, \qquad r \geq j + 3/2 \quad (6.2)$$

where $j = 1/2, 3/2, 5/2, \ldots$, $r = 2, 4, 6, \ldots$. and the rest of the elements are equal to zero

$$\gamma^{\dot{1}1} = \gamma_j^{12} = \gamma_j^{34} = \cdots = 0 \qquad \gamma^{1\dot{1}} = \gamma^{\dot{1}\dot{2}} = \gamma^{\dot{3}\dot{4}} = \cdots = 0\,. \quad (6.3)$$

The Lorentz boost operators $\vec{b}$ are antihermitian in this case

$$b_k^+ = -b_k\,, \quad (6.4)$$

because the amplitudes ς (2.10) are pure imaginary and therefore the Γ_k matrices are also antihermitian

$$\Gamma_k^+ = -\Gamma_k\,. \quad (6.5)$$

The matrix Ω changes and is now equal to the parity operator P, the relation

$$\Omega \Gamma_\mu^+ = \Gamma_\mu \Omega$$

remains valid. The diagonal part of Γ_k anticommutes with Γ_0 as it was before (4.22)

$$\{\Gamma_0, \tilde{\Gamma_k}\} = 0 . \qquad k = x, y, z \tag{6.6}$$

The mass spectrum is equal to

$$M_n^2 = \frac{2M^2}{n} \frac{(j+n)(j+n+1)}{j+(n+1)/2} \tag{6.7}$$

where $n = 1, 2, 3, ..$ and enumerates the trajectories (see fig. 6). The lowest spin on a given trajectory is either 1/2 or 3/2 depending on n: if n is odd then $j_{min} = 1/2$, if n is even $j_{min} = 3/2$. This is an essential new property of the dual equation because now we have an infinite number of states with a given spin j instead of $j + 1/2$. The string tension is the same as in the dual system (4.28)

$$2\pi\sigma_n = \frac{2M^2}{n} \qquad n = 1, 2, 3, \ldots \tag{6.8}$$

and the lower mass on a given trajectory n is given by the formula ($j = 1/2$)

$$M_n^2(j = 1/2) = \frac{4M^2}{n} \frac{(2n+1)(2n+3)}{n+2} \to (4M)^2 . \tag{6.9}$$

Thus the main problem of decreasing spectrum has been solved after the dual transformation because the spectrum is now bounded from below.

Including the Γ_5 mass term one can see that all trajectories acquire a nonzero slope

$$M_n^2 = \frac{2M^2}{n} \frac{(j+n)^2(j+n+1)^2}{j+(n+1)/2} \frac{1}{4} \tag{6.10}$$

where

$$n = 1, 2, 3, \ldots$$

$$j(n)_{min} = 1/2, 3/2, 1/2, \ldots$$

thus for the lower mass on a given trajectory n we have

$$M_n^2(j = 1/2) = \frac{M^2}{16} \frac{(2n+1)^2(2n+3)^2}{n(n+2)} \to M^2 n^2 . \tag{6.11}$$

Turning on the pure Casimir mass term $gM(\vec{a}^2 - \vec{b}^2)$ the spectrum will grows as j^5

$$M_n^2 = g^2 \frac{2M^2}{n} \frac{(j+n)^3(j+n+1)^3}{j+(n+1)/2} , \tag{6.12}$$

where $n = 1, 2, 3 \ldots$

7. Extension to bosons

Irrespective of the future interpretation of the theory it is important to include bosons. The appropriate representation Θ for this purpose is

$$\cdots\ (\lambda;-3)\ (\lambda;-2)\ (\lambda;-1)\ (\lambda;0)\ (\lambda;0)\ (\lambda;1)\ (\lambda;2)\ (\lambda;3)\ \ldots \tag{7.1}$$

and we shall consider $\lambda = 0$. The solution for Γ_0 is of the same form as we had before for fermions

$$\gamma_j^{r+1r} = \gamma_j^{rr+1} = \gamma_j^{\dot{r+1}\dot{r}} = \gamma_j^{\dot{r}\dot{r+1}} = \sqrt{\left(\frac{1}{4} - \frac{j^2+j}{4r(r-1)}\right)}\,, \qquad r \geq j+2 \tag{7.2}$$

where $j = 0,1,2,3,\ldots$, $r = 2,4,6,\ldots$ and the rest of the elements are equal to zero

$$\gamma^{\dot{1}1} = \gamma_j^{12} = \gamma_j^{34} = \cdots = 0 \gamma^{1\dot{1}} = \gamma^{\dot{1}\dot{2}} = \gamma^{\dot{3}\dot{4}} = \cdots = 0\,. \tag{7.3}$$

One can compute the slope of the first trajectory

$$M_1^2 = 2M^2(j+2) \approx 2M^2 j\,, \qquad j = 0,2,4,\ldots \tag{7.4}$$

thus for this trajectory the string tension σ is equal to

$$2\pi\sigma_1 = \frac{1}{\alpha_1'} = 2M^2 \tag{7.5}$$

and coincides with the one we obtained for the fermion string. The general formula for the mass spectrum is

$$M_n^2 = \frac{2M^2}{n}\ \frac{(j+n)(j+n+1)}{j+(n+1)/2} \qquad n = 1,2,3,\ldots \tag{7.6}$$

and the lower spin on a given trajectory is either 0 or 1 depending on n: if n is odd then $j_{min} = 0$, if n is even then $j_{min} = 1$. Comparing this spectrum with the fermionic one (6.7) one can see that fermions and bosons lie on the same trajectories. In the bosonic case we do not have Γ_5-matrix and the Casimir operator $(\vec{a}\cdot\vec{b})$ is zero because $\lambda = 0$, but we can include pure Casimir mass term $(\vec{a}^2 - \vec{b}^2)$ into the string equation to receive additional contribution to the slope of the trajectories. The formula for the spectrum is the same as for the fermions (6.12).

8. Discussion

In this article a possible relativistic equation for the gonihedric string which has the Dirac form was suggested. We did not have direct derivation of the above equation from the gonihedric string, which was formulated as a model of random surfaces, but the basic properties of the gonihedric string provide strong evidence for the existance of an appropriate equation. The equation has its own value independent of the motivation advocated in this article.

It is suggested to search the solutions of the Majorana commutation realations in the form of Jacoby matrices. Because Jacoby matrices have very specific form, the original equations factorize into separate equations for every element γ_j^{rr+1} and solutions therefore depend from $(4N-2)$ *j-independent free constant*. We essentially use this freedom to impose necessary physical constraints on a solutions requiring correct behaviour of the spectrum, Hermitian property of Γ_0, commutation relation with the matrix of invariant form and the positivity of the current density matrix. In general these are very complicated problems simply because one should study the spectral properties of matrices of infinite size. The solution which appear has exeptional behaviour: half of the eigenvalues of the spectrum are increasing and the other half are decreasing. One can achieve this exeptional behaviour by tuning the free constants in the general solution. One should take into account that matrix elements γ_j^{rr+1} have very comlicated "root" dependence on j and this fact makes complite investigation of the spectrum extremely complicated, but one can manage to solve the problem exactly.

Thus half of the eigenvalues (decreasing eigenvalues), produce quasilinear trajectories with nonzero string tension $\sigma_n = 2M^2/n$ (1.6) and the other half (increasing eigenvalues) affect the low spin states on trajectories, so that smallest masses on a given trajectory tends to zero (4.30).

This last problem has been solved by dual transformation. We have found a *new exact symmetry* which transforms two solutions of the Majorana commutation relations one into another. Indeed, interchanging j_0 and λ in the representation $\Theta = (j_0, \lambda)$ does not affect the matrix elements of the Lorentz generators $I_{\mu\nu}$ therefore a solution γ_j for $\Theta = (j_0, \lambda)$ can be translated into a solution γ_j^{dual} for the dual representation Θ^{dual} by exchanging j_0 for λ and leting spin j to run in a different interval $j = \lambda, \lambda+1, \ldots$. Thus the dual transformation does not change the actual j dependence of γ_j, but what is important here is that despite the fact that the dual solution γ_j^{dual} is almost identical with γ_j (we mean the j dependence), the spectrum essentially changes for the low spin statets and does not affect the high spin states. This is because the number of states with spin j, which was equal to $j + 1/2$ before the dual transformation become infinite now. Indeed for higher spin states, where the matrix γ_j was already very large, the mass spectrum remains identical and increasing, while for low spin states, where the matrix γ_j was of a small size, now becomes also infinite-dimensional and the mass spectrum is bounded from below. All $B-H-\Sigma$-solutions and there duals $B^{dual} - H^{dual} - \Sigma^{dual}$ form an $exact\ family$ of intimate solutions.

Thus the spectrum of the theory consists of particles and antiparticles of increasing half-integer spin lying on quasilinear trajectories of different slope. Explicit formulas for the mass spectrum allow to compute the string tension and thus demonstrate the string character of the theory. The old problem of decreasing mass spectrum has been solved and the spectrum is bounded from below. The trajectories are only asymptotically linear and thus are different from the free string linear trajectories. It is difficult to say at the moment what is the physical reason for this nonperturbative behaviour. Nonzero Casimir operators and the generalization of the Γ_5 matrix allow to introduce additional mass terms into the string equation and to increase the slope of the trajectories.

The equation is explicitly Lorentz invariant, but has unwanted ghost solutions. The tachyonic solutions which appear in Majorana equation do not show up here. This is because the nondiagonal transition amplitudes of the form $< \cdots j \cdots |\Gamma_k| j \pm 1 >$ are small here and the diagonal amplitudes $< \cdots j \cdots |\Gamma_k| j >$ are large. Indeed in the Majorana equation the diagonal elements are equal to zero, therefore the nondiagonal elements give rise to space-like tachyonic solutions (see (20) in [12]). In the Dirac equation only diagonal transition amplitudes are present, and so the equation does not admit tachyonic solutions. The problem of ghost states is more subtle here and we shall analyze the natural constraints appearing in the system to ensure that they decouple from the physical space of states.

Acknowledgments

In conclusion one of us (G.K.S.) wishes to acknowledge the hospitality of the Niels Bohr Institute where this work was started and would like to thank J.Ambjorn, P.Olesen, G.Tiktopoulos and P.Di Vecchia for the interesting discussions. We are thankful to Pavlos Savvidis, who has found the recurrent relation for the computation of the characteristic polynomials of the Jacoby matrices which finally leads to the basic solution for Γ_0. This work has been initiated during Triangular Meeting in Rome, in March 1996, where we got the collected articles of Ettore Majorana: "La vita e l'opera di Ettore Majorana (1906-1938)" Roma, Accademia Nazionale dei Lincei, 1966, (ed. by E. Amaldi).

A. Jacoby matrices

Jacoby matrices. We are searching for the solution of the equation (2.42)

$$\Gamma_0 b_z^2 - 2 b_z \Gamma_0 b_z + b_z^2 \Gamma_0 = -\Gamma_0$$

in the form of Jacoby matrices [16]

$$\gamma_j = \begin{pmatrix} 0, & \gamma_j^{NN-1}, & & & \\ \gamma_j^{N-1N}, & 0, & \gamma_j^{N-1N-2}, & & \\ & \ddots & \ddots & \ddots & \\ & & \gamma_j^{\dot{N}-1\dot{N}-2}, & 0, & \gamma_j^{\dot{N}-1\dot{N}} \\ & & & \gamma_j^{\dot{N}\dot{N}-1}, & 0 \end{pmatrix}.$$

The nonzero matrix elements are given by the formulas (2.5), (2.14), (2.15) and (2.41). To get equations for $\gamma^{rr'}_j$ one should compute the projection of the above matrix equation to the states $< j \pm 1| \quad |j \mp 1 >$, $< j| \quad |j >$ and $< j||j-1 >$, $< j-1||j >$. These equations are $\gamma^{rr'}_{j-1}\varsigma^{r'}_j\varsigma^{r'}_{j+1} - 2\varsigma^r_j\gamma^{rr'}_j\varsigma^{r'}_{j+1} + \varsigma^r_j\varsigma^r_{j+1}\gamma^{rr'}_{j+1} = 0$ and $\gamma^{rr'}_{j-1}\varsigma^{r'}_j(\lambda^{r'}_j - 2\lambda^r_{j-1} + \lambda^{r'}_{j-1}) + \gamma^{rr'}_j\varsigma^r_j(\lambda^r_j - 2\lambda^{r'}_j + \lambda^r_{j-1}) = 0$ for the nondiagonal transitions (plus their transpose). For diagonal elements they are

$$\begin{aligned}\gamma^{rr'}_j[(j+1)^2((\varsigma^{r'}_{j+1})^2 + (\varsigma^r_{j+1})^2) + j^2((\varsigma^{r'}_j)^2 + (\varsigma^r_j)^2)]& \\ -2(j+1)^2\varsigma^r_{j+1}\gamma^{rr'}_{j+1}\varsigma^{r'}_{j+1} - 2j^2\varsigma^r_j\gamma^{rr'}_{j-1}\varsigma^{r'}_j &= -\gamma^{rr'}_j\end{aligned}$$

together with

$$\gamma^{rr'}_j[(\varsigma^{r'}_{j+1})^2 + (\varsigma^r_{j+1})^2 + (\varsigma^{r'}_j)^2 + (\varsigma^r_j)^2 - (\lambda^{r'}_j - \lambda^r_j)^2] - 2\varsigma^r_{j+1}\gamma^{rr'}_{j+1}\varsigma^{r'}_{j+1} - 2\varsigma^r_j\gamma^{rr'}_{j-1}\varsigma^{r'}_j = 0\,.$$

For the Majorana representations [12, 14] it is easy to see that these equations reduces to $\gamma_{j+1} - 2\gamma_j + \gamma_{j-1} = 0$ and $(2j+1)(\gamma_{j+1} - \gamma_j) = 2\gamma_j$ with the solution $\gamma_j = Const(j+1/2)$. The $N = 1$ case can be solved using Jacoby parametrization given above, with the result $\gamma^{1\dot{1}}_j = a(j+1/2)$ and $\gamma^{\dot{1}1}_j = b(j+1/2)$, where a and b are arbitrary j-independent constants. Because every element of the Jacoby matrix γ^{rr+1}_j should separately sutisfy these equations one can directly check that

$$\gamma^{rr+1}_j = Const\sqrt{(1 - \frac{r^2}{N^2})} \cdot \sqrt{(\frac{j^2+j}{4r^2-1} - \frac{1}{4})}$$

is a solution of the above equations. The remaining j-independent constant was determined in the main text by requiring the appropriate physical properties of the solution. For that one should solve eigenvalue problem for the matrix $\gamma^{rr'}_j$ of increasing complexity (see section 5). Useful formulas are

$$\begin{pmatrix} 0, & a \\ a, & 0 \end{pmatrix}, \quad \begin{pmatrix} 0, & b, & & \\ b, & 0, & a, & \\ & a, & 0, & b \\ & & b, & 0 \end{pmatrix}, \quad \begin{pmatrix} 0, & c, & & & & \\ c, & 0, & b, & & & \\ & b, & 0, & a, & & \\ & & a, & 0, & b, & \\ & & & b, & 0, & c \\ & & & & c, & 0 \end{pmatrix} \ldots. \tag{A.1}$$

The characteristic polynomials have the form

$$\begin{aligned} x^2 - a^2 &= 0 \\ x^4 - (a^2 + 2b^2)x^2 + b^4 &= 0 \\ x^6 - (a^2 + 2b^2 + 2c^2)x^4 + (2a^2c^2 + 2b^2c^2 + b^4 + c^4)x^2 - a^2c^4 &= 0 \\ x^8 - (a^2 + 2b^2 + 2c^2 + 2d^2)x^6 + & \qquad (A.2) \\ (2a^2c^2 + 2b^2c^2 + 2a^2d^2 + 2c^2d^2 + b^4 + c^4 + d^4 + 4b^2d^2)x^4 - & \\ -(a^2c^4 + a^2d^4 + 2b^2d^4 + 2b^4d^2 + 2a^2c^2d^2 + 2b^2c^2d^2)x^2 + b^4d^4 &= 0 \\ \cdots\cdots\cdots & \qquad (A.3) \end{aligned}$$

and so on.

B. σ and τ algebra

σ and τ algebra. For **j=1/2** we have $\sigma_{1/2} = 0$. **j=3/2** For this spin we have $\sigma_{3/2}, \sigma^*_{3/2}, \sigma^2_{3/2} = \sigma^{*2}_{3/2} = 0$ and the main relation is

$$\sigma_{3/2}\sigma^*_{3/2} + \sigma^*_{3/2}\sigma_{3/2} = 4 \tag{B.1}$$

and thus $\gamma_{3/2} = \omega_{3/2}(1 + \sigma_{3/2})$. **j=5/2** For this spin we have $\sigma_{5/2}, \sigma^*_{5/2}, \sigma^2_{5/2}, \sigma^{*2}_{5/2}$, $\sigma^3_{5/2} = \sigma^{*3}_{5/2} = 0$ which appear to satisfy the algebra

$$\{\sigma_{5/2}, \sigma^*_{5/2}\} - 2(\sigma_{5/2} + \sigma_{5/2}) - 3\{\sigma^2_{5/2}, \sigma^{*2}_{5/2}\} + 4(\sigma^2_{5/2} + \sigma^{*2}_{5/2}) = 2 \tag{B.2}$$

Then it follows that $\gamma_{5/2} = \omega_{5/2}(1 + \sigma_{5/2})$ and that

$$\sigma^2_{5/2}\sigma^{*2}_{5/2}\sigma^2_{5/2} = 2\sigma^2_{5/2}\sigma^*_{5/2}\sigma^2_{5/2}\,. \tag{B.3}$$

Similar algebra takes place for the τ_j matrices.

C. Parity and Γ_5 matrices

*Parity and Γ_5 matrices.*The commutation relations which define the parity operator P are

$$Pa_k = a_k P \qquad Pb_k = -b_k P \qquad P^2 = 1\,. \tag{C.1}$$

Since the parity operator P commutes with spatial rotations we have

$$P = P_j^{rr'}\delta_{jj'}\delta_{mm'} \tag{C.2}$$

and from the second commutator

$$P_j^{r\dot{r}} = -P_{j-1}^{r\dot{r}} \qquad P_j^{\dot{r}r} = -P_{j-1}^{\dot{r}r}\,, \tag{C.3}$$

thus

$$P_j^{r\dot{r}} = P_j^{\dot{r}r} = (-1)^{[j]} \tag{C.4}$$

One can check that

$$P\Gamma_0 = \Gamma_0 P \qquad P\Gamma_k = -\Gamma_k P \qquad P\Omega = \Omega P \tag{C.5}$$

$$P_{1/2} = \begin{pmatrix} 0, 1 \\ 1, 0 \end{pmatrix}, \quad P_{3/2} = \begin{pmatrix} & & & -1 \\ & & -1 & \\ & -1 & & \\ -1 & & & \end{pmatrix},$$

$$P_{5/2} = \begin{pmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \\ & 1 & & & & \\ 1 & & & & & \end{pmatrix}, \ldots \tag{C.6}$$

For Γ_5 we have

$$\Gamma_5 a_k = a_k \Gamma_5 \qquad \Gamma_5 b_k = b_k \Gamma_5 \qquad \Gamma_5^2 = 1 \tag{C.7}$$

because it commutes with spatial rotations we have

$$\Gamma_5 = \Gamma_{5j}^{rr'} \delta_{jj'} \delta_{mm'} \tag{C.8}$$

and from the second commutator

$$\Gamma_{5j}^{rr} = \Gamma_{5j-1}^{rr} \qquad \Gamma_{5j}^{\dot{r}\dot{r}} = \Gamma_{5j-1}^{\dot{r}\dot{r}} \qquad \Gamma_{5j}^{rr} = -\Gamma_{5j}^{\dot{r}\dot{r}} \tag{C.9}$$

thus

$$\Gamma_{5j}^{rr} = -\Gamma_{5j}^{\dot{r}\dot{r}} = (-1)^{r+1} \,. \tag{C.10}$$

One can check that

$$\Gamma_5 \Gamma_0 = -\Gamma_0 \Gamma_5 \qquad \Gamma_5 \Gamma_k = -\Gamma_k \Gamma_5 \qquad \Gamma_5 P = -P \Gamma_5 \qquad \Gamma_5 \Omega = -\Omega \Gamma_5 \tag{C.11}$$

and thus

$$\Gamma_{51/2} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \qquad \Gamma_{53/2} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix},$$

$$\Gamma_{55/2} = \begin{pmatrix} 1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & 1 & \\ & & & & & -1 \end{pmatrix}, \qquad \ldots \tag{C.12}$$

D. Space gamma matrices

One can compute Γ_k matrices using relation $\Gamma_k = i(b_k \Gamma_0 - \Gamma_0 b_k)$, $k = x, y, z$

$$\begin{aligned}
< j, m, r|\Gamma_z|r+1, j, m > &= im(\lambda_j^r - \lambda_j^{r+1}) \cdot \gamma_j^{rr+1} \\
< j, m, r+1|\Gamma_z|r, j, m > &= -im(\lambda_j^r - \lambda_j^{r+1}) \cdot \gamma_j^{r+1r} \\
< j-1, m, r|\Gamma_z|r+1, j, m > &= i\sqrt{j^2 - m^2} \cdot (\varsigma_j^r \gamma_j^{rr+1} - \gamma_{j-1}^{rr+1} \varsigma_j^{r+1}) \\
< j-1, m, r+1|\Gamma_z|r, j, m > &= i\sqrt{j^2 - m^2} \cdot (\varsigma_j^{r+1} \gamma_j^{r+1r} - \gamma_{j-1}^{r+1r} \varsigma_j^r) \\
< j, m, r+1|\Gamma_z|r, j-1, m > &= -i\sqrt{j^2 - m^2} \cdot (\varsigma_j^r \gamma_j^{r+1r} - \gamma_{j-1}^{r+1r} \varsigma_j^{r+1}) \\
< j, m, r|\Gamma_z|r+1, j-1, m > &= -i\sqrt{j^2 - m^2} \cdot (\varsigma_j^{r+1} \gamma_j^{rr+1} - \gamma_{j-1}^{rr+1} \varsigma_j^r) \,.
\end{aligned} \tag{D.1}$$

It is appropriate to introduce separate notations for diagonal and nondiagonal parts of Γ_k, we shall define them as $\tilde{\Gamma}_k$ and $\tilde{\tilde{\Gamma}}_k$.

References

[1] H.B. Nielsen and P. Olesen, *Phys. Lett.* **B 32** (1970) 203;
D.J. Gross and F. Wilczek, *Phys. Rev. Lett.* **30** (1973) 1343;
H.D. Politzer, *Phys. Rev. Lett.* **30** (1973) 1346;
K. Wilson, *Phys. Rev.* **D 10** (1974) 2445;
G. 't Hooft, *Nucl. Phys.* **B 72** (1974) 461; *ibid.* **75** (1975) 461;
G.K. Savvidy, *Phys. Lett.* **B 71** (1977) 133;
A. Polyakov, *Gauge Fields and String*, Harwood Academic Publishers, 1987;
D. Gross and W. Taylor, *Nucl. Phys.* **B 400** (1993) 181 [`hep-th/9301068`]; *ibid.* **403** (1993) 395;
N. Seiberg and E. Witten, *Nucl. Phys.* **B 426** (1994) 19 [`hep-th/9407087`]; *ibid.* **431** (1994) 484.

[2] G.K. Savvidy and K.G. Savvidy, *Gonihedric string and asymptotic freedom*, *Mod. Phys. Lett.* **A 8** (1993) 2963 [`hep-th/9301001`];
R.V. Ambartzumian, G.K. Savvidy , K.G. Savvidy and G.S. Sukiasian *Phys. Lett.* **B 275** (1992) 99;
G.K. Savvidy and K.G. Savvidy *Int. J. Mod. Phys.* **A 8** (1993) 3993 [`hep-th/9208041`].

[3] B. Durhuus and T. Jonsson, *Phys. Lett.* **B 297** (1992) 271 [`hep-th/9209094`].

[4] G.K. Savvidy and R. Schneider, *Comm. Math. Phys.* **161** (1994) 283 [`hep-th/9401045`].

[5] G.K. Savvidy and F.J. Wegner, *Nucl. Phys.* **B 413** (1994) 605 [`hep-th/9308094`].

[6] G.K. Savvidy and K.G. Savvidy, *Phys. Lett.* **B 324** (1994) 72 [`hep-lat/9311026`].

[7] G.K. Savvidy and K.G. Savvidy, *Interaction hierarchy*, *Phys. Lett.* **B 337** (1994) 333 [`hep-th/9409030`]; *Mod. Phys. Lett.* **A 11** (1996) 1379 [`hep-th/9506184`].

[8] J. Ambjorn, B. Durhuus and T. Jonsson, *J. Phys.*, **A 21** (1988) 981.

[9] R. Pietig and F.J. Wegner, *Nucl. Phys.* **B 466** (1996) 513 [`hep-th/9604013`].

[10] G.Koutsoumbas, G.K. Savvidy and K.G. Savvidy, *Phase structure of four-dimensional gonihedric spin system*, *Phys. Lett.* **B 410** (1997) 214 [`hep-th/9706173`] [`hep-th/9706173`];
G.K. Bathas *et al.*, *Mod. Phys. Lett.* **A 10** (1995) 2695 [`hep-th/9504054`];
M. Baig, D. Espriu, D. Johnson and R.K.P.C. Malmini, *J. Phys.*, **A 30** (1997) 407; *ibid.* **A 30** (1997) 7695.

[11] P.A.M. Dirac, *The Quantum Theory of Electron, Proc. Roy. Soc.*, **A 117** (1928) 610; *ibid.* **A118** (1928) 351.

[12] E. Majorana, *Teoria Relativistica di Particelle con Momento Intrinseco Arbitrario*, *Nuovo Cim.* **9** (1932) 335.

[13] P. Ramond, *Dual Theory for Free Fermions*, *Phys. Rev.* **D 3** (1971) 2415.

[14] P.A.M. Dirac, *Relativistic wave equations, Proc. Roy. Soc.* **A 155** (1936) 447; *Unitary Representation of the Lorentz Group, Proc. Roy. Soc.* **A 183** (1944) 284;
M. Fierz and W. Pauli, *On Relativistic Wave Equations for Particles of Arbitrary Spin in an Electromagnetic Field, Proc. Roy. Soc.* **A 173** (1939) 211;
W.Rarita and J.Schwinger, *On a Theory of Particles with Half-Integral Spin, Phys. Rev.* **60** (1941) 61;
I.M. Gel'fand and M.A. Naimark, *J. Phys. (USSR)* **10** (1946) 93;
I.M.Gel'fand and A.M. Yaglom, *Zh. Exp. Teor. Fiz.* **18** (1948) 707;
Y. Nambu, *Phys. Rev.* **160** (1967) 1171;
W. Rúhl, *Comm. Math. Phys.* **6** (1967) 312.

[15] J. Schwarz (Editor), *Superstrings: the First 15 Years of Superstring Theory*, Vols. **1,2**, World Scientific, 1985;
M. Green, J. Schwarz and E. Witten, *Superstring Theory*, Vols. **1,2**, Cambridge University Press, Cambridge, 1988;
J. Polchinski, *TASI Lectures on D-Branes*, NSF-ITP-96-145, `hep-th/9611050`;
T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *M-theory as a Matrix Model: A Conjecture*, `hep-th/9610043`;
C. Bachas, *Introduction to D-branes*, `hep-th/9701019`.

[16] G.K. Savvidy, *Symplectic and large-N gauge theories.* In *Cargese 1990, Proceedings, Vacuum Structure in Intense Fields*, Vol. **225**, Plenum Press, NY, 1991, p. 415;
Renormalization of infinite dimensional gauge symmetries (Minnesota U. and Yerevan Phys. Inst.). TPI-MINN-90-2-T, Jan 1990. 18pp.

[17] G.K. Savvidy and K.G. Savvidy, `hep-th/9711015`.

[18] M. Born, W. Heisenberg and P. Jordan, *Z. Physik* **C 35** (1926) 557;
H. Weyl, *Math. Z* **24** (1926) 342;
H. Weyl, *The Theory of Groups and Quantum Mechanics*, London, Methuen, 1931;
R. Brauer and H. Weyl, *Amer. J. Math.* **57** (1935) 425;
E. Cartan, *Bull. Soc. Math. France* **41** (1913) 53;
Van der Waerden, *Die gruppentheoretische methode der Quantummechanik*, Berlin, Springer, 1932; *Nachr. Ges. Wiss. Gott.* (1929) 100;
E.P. Wigner, *Ann. Math.* **40** (1939) 149; *Z. Physik* **C 124** (1948) 665;
O. Laporte and G. Uhlenbeck, *Phys. Rev.* **37** (1931) 1380.